

# HKUST - MATH3312, Fall 2019

## Final Examination (100 points, 120 minutes)

Please read the following instructions carefully:

- (1) You are allowed 120 minutes for this off-site open book exam. Please do NOT communicate with any other people during the exam period.
- (2) Please clearly mark your name and student ID in your answer sheets.
- (3) After finish the exam, you should scan all your answer sheets and generate them into one PDF file. The PDF should be named as: surname-forename.pdf . Please email the PDF back to TA (tjiangad@connect.ust.hk) in 5 minutes after the end of the exam.
- (4) To receive full credits, you MUST justify your answers with supporting reasonings. A correct answer with wrong or no reasonings will NOT receive any credits.
- (5) Use 5-(significant) digit rounding arithmetic in the major steps of all calculations.

**[points]**

1. Let  $f(x) = \sin(e^{-2x})$ .

(a) Use the two-point backward-difference and forward-difference formulas with  $h = 0.1$  to approximate  $f'(0.5)$ ;

**4 points**

(b) Use the three-point formula with  $x_0 = 0.4, x_1 = 0.5, x_2 = 0.6$  to approximate  $f'(0.6)$ .

**4 points**

2. Suppose  $f \in C^4[a, b]$  and  $x_0 \in (a, b)$ , and that  $h$  is small enough such that  $x_0 \pm h \in (a, b)$ . Use Taylor expansion to derive an approximation for  $f''(x_0)$  in terms of  $f(x_0 - h), f(x_0)$  and  $f(x_0 + h)$  with error in the order of  $O(h^2)$ .

**10 points**

3. (a) Use the Composite Simpson's rule with step  $h = 0.1$  to approximate the integral  $\int_1^2 e^{-x^2} dx$ .

**8 points**

(b) Use the Composite Trapezoidal rule with step  $h = 0.2$  to approximate the integral  $\int_1^2 e^{-x^2} dx$ .

**8 points**

4. Suppose  $f \in C^4[a, b]$  and  $x_0 \in (a, b)$ , and that  $h$  is small enough such that  $x_0 \pm h \in (a, b)$ . Derive Simpson's rule for approximating  $\int_{x_0-h}^{x_0+h} f(x)dx$  with the error term in the order of  $O(h^5)$ .

**16 points**

5. (a) Use Euler's method with step  $h = 0.1$  to approximate the solution  $y(t)$  at  $t = 1.3$  for the initial-value problem:

$$y' = -y + t^2 y^{1/2}, \quad 1 \leq t \leq 2, \quad \text{and} \quad y(1) = 2.$$

**10 points**

(b) Suppose  $f$  is continuous and satisfies a Lipschitz condition with constant  $L$  on  $D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$  and that a constant  $M$  exists with

$$|y''(t)| \leq M, \quad \forall t \in [a, b],$$

where  $y(t)$  denotes the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad \text{and} \quad y(a) = \alpha.$$

Let  $w_0, w_1, \dots, w_N$  be the approximations generated by Euler's method for some positive integer  $N$ , and let  $h = (b - a)/N$  and  $y_i = y(a + ih)$ ,  $i = 0, 1, 2, \dots, N$ . Derive an upper bound for  $|y_{i+1} - w_{i+1}|$  in terms of  $|y_i - w_i|$ ,  $h$ ,  $L$  and  $M$ , for  $i = 0, 1, \dots, N - 1$ .

**15 points**

6. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

(a) Use Gaussian elimination with partial pivoting to find the solution of  $Ax = b$ ;

**10 points**

(b) Use Gaussian elimination without row interchange to find the lower triangular matrix  $L$  with all the diagonal entries being 1 and upper triangular matrix  $U$  such that  $A = LU$ , and then use the forward and backward substitution algorithms to solve  $Ax = b$ .

**15 points**

— End —